ERGODIC DYNAMICS OF THE COUPLED QUASIGEOSTROPHIC FLOW-ENERGY BALANCE SYSTEM

AIJUN DU, JINQIAO DUAN, HONGJUN GAO, AND TAMAY ÖZGÖKMEN

ABSTRACT. The authors consider a mathematical model for the coupled atmosphere-ocean system, namely, the coupled quasigeostrophic flow-energy balance model. This model consists of the large scale quasigeostrophic oceanic flow model and the transport equation for oceanic temperature, coupled with an atmospheric energy balance model. After reformulating this coupled model as a random dynamical system (cocycle property), it is shown that the coupled quasigeostrophic-energy balance fluid system has a random attractor, and under further conditions on the physical data and the covariance of the noise, the system is ergodic, namely, for any observable of the coupled atmosphere-ocean flows, its time average approximates the statistical ensemble average, as long as the time interval is sufficiently long.

1. MATHEMATICAL MODEL

We consider large scale geophysical flows modeled by the quasigeostrophic flow equation in the horizontal $xy-$plane, in terms of vorticity $q(x, y, t)$, and the transport equation for the oceanic temperature $T(x, y, t)$, coupled with the atmospheric energy balance equation proposed by North and Cahalan [16] for the air temperature $\Theta(x, y, t)$, on the domain $D = \{(x, y) : 0 \leq x \leq l, 0 \leq y \leq l\}$:

$$\Theta_t = \Delta\Theta - (a + \Theta) + S_a(x, y) - b(y)(S_o(x, y) + \Theta - T(x, y)) + \dot{w},$$

$$(1) \quad \frac{\partial q}{\partial t} = \nu\Delta q - rq + Pr\, Ra\, \partial_y T - J(\psi, q + \beta y),$$

$$\frac{\partial T}{\partial t} = \Delta T - J(T, \psi),$$

where $\psi(x, y, t)$ is the stream function, $\beta \geq 0$ is the meridional gradient of the Coriolis parameter, $\nu > 0$ is the viscous dissipation constant, and $r > 0$ is the Ekman dissipation constant. Furthermore,

$$q(x, y, t) = \Delta\psi(x, y, t)$$

is the vorticity, $a > 0$ is a constant parameterizing the effect of the earth's longwave radiative cooling, $b(y)$ is the latitudinal fraction of the earth covered by the ocean basin, Pr is the Prandtl number, and Ra is the Rayleigh number. Note that $S_a(x, y)$ and $S_o(x, y)$ are empirical functions representing the effects (on atmosphere and ocean, respectively) of the shortwave solar radiation. Moreover, $J(g, h) = g_x h_y - g_y h_x$ is the Jacobian operator and $\Delta = \partial_{xx} + \partial_{yy}$ is the Laplacian operator. All these equations are in non-dimensionalized forms. The fluctuating noise $\dot{w}(x, y, t)$ is usually of a shorter time scale than the response time scale of

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the air mean temperature. So we neglect the autocorrelation time of this fluctuating process and thus assume that the noise is white in time. The spatially correlated white-in-time noise $\dot{w}(x, y, t)$ is described as the generalized time derivative of a Wiener process $w(x, y, t)$ defined in a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, with mean zero and covariance operator Q. For geophysical background of similar coupled atmosphere-ocean models, see [8, 2]. Recently, a few authors have considered the randomly forced quasigeostrophic equation, in order to incorporate the impact of uncertain geophysical forces ([13], [14], [7]).

The fluid boundary condition is no normal flow and free-slip on the whole boundary:

$$(2) \qquad\qquad\qquad \psi = 0, \; q = 0.$$

The flux boundary conditions are assumed for the ocean temperature T and air temperature Θ:

$$(3) \qquad\qquad\qquad \frac{\partial \Theta}{\partial n} = \frac{\partial T}{\partial n} = 0,$$

with n is the outer unit normal vector on boundary. By equation (1) and the boundary condition, we find that $\int_D T(x, y) dx dy =$ constant, and thus we assume that $\int_D T(x, y) dx dy = 0$ without loss of generality.

This is a coupled system of both deterministic and stochastic partial differential equations.

This paper is organized as follows. In the next section, we introduce basic concepts in random dynamical systems. Then we reformulate the coupled fluid system (1) as a random dynamical system with help of a Ornstein-Uhlenbeck random stationary process in §3. After obtaining basic estimates for the system in §4, we show in §5 that the coupled fluid system (1) admits a random attractor. Under further conditions on the physical data and the covariance of the noise, we show that the system is actually ergodic, namely, for any observable of the coupled atmosphere-ocean flows, its time average approximates the statistical ensemble average, as long as the time interval is sufficiently long.

2. Random dynamical systems

In order to investigate the long time dynamics of the coupled fluid system (1) under the influence of random forces, we need some appropriate concepts and tools from the theory of *random dynamical systems*.

A random dynamical system consists of two components. The first component is a *driven flow* $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$ as a model for noise, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and θ is a $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}), \mathcal{F}$ measurable flow: we have

$$\theta_0 = \mathrm{id}, \qquad \theta_{t+\tau} = \theta_t \circ \theta_\tau =: \theta_t \theta_\tau$$

for $t, \tau \in \mathbb{R}$. To express that the noise is stationary and "chaotic", the measure $\mathbb{P}$ is supposed to be ergodic with respect to θ. The second component of a random dynamical system is a $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F} \otimes \mathcal{B}(H), \mathcal{B}(H)$-measurable mapping φ satisfying the *cocycle* property

$$\varphi(t + \tau, \omega, x) = \varphi(t, \theta_\tau \omega, \varphi(\tau, \omega, x)), \qquad \varphi(0, \omega, x) = x,$$

where the phase space H is a separable metric space and x is chosen arbitrarily in H. We will denote this random dynamical system by symbol φ.

A standard model for a spatially correlated white-in-time noise is the generalized time derivative of a two-sided *Brownian motion or Wiener process* $w(x, y, t)$. Let U be a separable Hilbert space with scalar product $(\cdot, \cdot)$ and the induced norm $\|\cdot\|$. For stochastic partial differential equations containing such a noise, an appropriate or canonical probability space is

$$(C_0(\mathbb{R}, U), \mathcal{B}(C_0(\mathbb{R}, U)), \mathbb{P}),$$

where $C_0(\mathbb{R}, U)$ is the space of continuous functions on $\mathbb{R}$, which take zero value at time zero. This space is given the compact-open topology (i.e., uniform convergence on compact intervals in $\mathbb{R}$). This topology is metrizible as it can be generated by the complete metric

$$d(g_1, g_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(g_1, g_2)}{1 + d_n(g_1, g_2)},$$

where $d_n(g_1, g_2) = \max_{|t| \leq n} \|g_1(t) - g_2(t)\|$, for g_1, g_2 in $C_0(\mathbb{R}, U)$. Thus we have open balls or open sets in $C_0(\mathbb{R}, U)$, and $\mathcal{B}(C_0(\mathbb{R}, U))$ is the corresponding Borel σ-algebra. Suppose the Wiener process w has covariance operator Q on U. Let $\mathbb{P}$ denote the *Wiener measure* with respect to Q. Note that $\mathbb{P}$ is ergodic with respect to the *Wiener shift* θ_t:

$$(4) \qquad \theta_t \omega = \omega(\cdot + t) - \omega(t), \qquad \text{for } \omega \in C_0(\mathbb{R}, U).$$

A major example of a random dynamical system is a random differential equation. For example, let us consider the following evolution equation in some Hilbert space

$$(5) \qquad \frac{du}{dt} = f(u, \theta_t \omega), \quad u(0) = x,$$

over some metric dynamical system $(\Omega, \mathcal{F}, \mathbb{P}, \theta)$. If (5) is well-posed for every $\omega \in \Omega$ and solutions $u(t, \omega; x)$ depend measurably on (t, ω, x), then the operator

$$\varphi \; : \; (t, \omega, x) \to u(t, \omega; x)$$

defines a random dynamical system (cocycle) φ. For detailed presentation of random dynamical systems we refer to the monograph by Arnold [1].

Motivated by deterministic dynamical systems we introduce several useful concepts from the theory of random dynamical systems.
A closed set $B(\omega)$, depending on ω, in a separable Hilbert space H is called random if the distance mapping $\omega \to \sup_{x \in B(\omega)} \|x - y\|_H$ is a random variable for any $y \in H$. A random dynamical system is called *dissipative* if there exists a random set B that is bounded for any ω and that is absorbing: for any random variable $x(\omega) \in H$ there exists a $t_x(\omega) > 0$ such that if $t \geq t_x(\omega)$, then

$$\varphi(t, \omega, x(\omega)) \in B(\theta_t \omega).$$

In the deterministic case (φ is independent of ω) the last relation coincides with the definition of an absorbing set. In the case of partial differential equations of *parabolic type*, due to the smoothing property, it is usually possible to prove that a dissipative system possesses compact invariant absorbing sets. For more details, see Temam [23], page 22f. Hence for a system of stochastically forced *parabolic* partial differential equations, such as the stochastic two-layer fluid system introduced in the

last section, we usually consider the random set $B(\omega)$ to be compact. In addition, we will assume that $B(\omega)$ is forward invariant:

$$\varphi(t, \omega, B(\omega)) \subset B(\theta_t \omega), \ t > 0.$$

In the following we also need a concept of *tempered random variables*. A random variable x is called tempered if

$$t \to |x(\theta_t \omega)|$$

is subexponentially growing:

$$\limsup_{t \to \pm\infty} \frac{\log^+ |x(\theta_t \omega)|}{|t|} = 0 \quad \text{a.s.}$$

where

$$log^+(x) = max\{0, log(x)\}$$

This technical condition is not a very strong restriction because the only alternative is that the above $\limsup$ is ∞, which describes the degenerate case of stationarity; see Arnold [1], page 164 f.

3. Cocycle property

In this section we will show that the coupled fluid model (1) defines a random dynamical system (cocycle property). The cocycle property essentially comes from the well-posedness.

Now we re-formulate the model such that appropriate tools of the theory of random dynamical system can be applied to analyse the coupled atmosphere-ocean model under a random wind forcing. For the following we need some tools from the theory of partial differential equations.

Let $H^1(D)$ be the Sobolev space of functions on D with first generalized derivative in $L_2(D)$, the function space of square integrable functions on D with norm and inner product

$$\|u\|_{L_2} = \left(\int_D |u(x)|^2 dD\right)^{\frac{1}{2}}, \quad (u, v)_{L_2} = \int_D u(x)v(x)dD, \quad u, v \in L_2(D).$$

The space $H^1(D)$ is equipped with the norm

$$\|u\|_{H^1} = \|u\|_{L_2} + \|\partial_y u\|_{L_2} + \|\partial_z u\|_{L_2}.$$

Motivated by the zero-boundary conditions of q we also introduce the space $\overset{\circ}{H}{}^1(D)$ which contains roughly speaking functions which are zero on the boundary ∂D of D. This space can be equipped with the norm

(6)
$$\|u\|_{\overset{\circ}{H}{}^1(D)} = \|\partial_y u\|_{L_2} + \|\partial_z u\|_{L_2}.$$

Another Sobolev space is given by $\dot{H}^1(D)$ which is a subspace of $H^1(D)$ consisting of functions u such that $\int_D u \, dx dy = 0$. A norm equivalent to the H^1-norm on $\dot{H}^1(D)$ is given by the right hand side of (6). For functions in $L_2(D)$ having this property we will write $\dot{L}_2(D)$.

For convenience, we introduce the vector notation for unknown geophysical quantities $u = (\Theta, q, T)$

We now take the linear differential operator from (1)

$$A(u) = \begin{pmatrix} -\Delta\Theta + (1 + b(x,y))\Theta \\ -\nu\Delta q \\ -\Delta T \end{pmatrix}$$

Recall that the function $0 < b(y) < 1$. A is defined on functions that are sufficiently smooth. We also have the following boundary conditions from §.2

$$\frac{\partial\Theta}{\partial n} = \frac{\partial T}{\partial n} = 0,$$

$$\psi|\partial D = 0, \ q|\partial D = 0.$$

We introduce the phase space for our geophysical quantities

$$H = L_2(D) \times L_2(D) \times L_2(D)$$
$$V = H^1(D) \times \overset{\circ}{H}{}^1(D) \times \dot{H}^1(D).$$

After this preparation, we are able to write our problem as a stochastic evolution equation.

Let $\dot{w}$ be a noise on $L_2(D)$ with finite energy given by the covariance operator Q of the Wiener process $w(t)$ which is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For the vector

$$W = (w, 0, 0),$$

we rewrite the coupled atmosphere-ocean system (1) as a stochastic evolution equation on V':

$$(7) \qquad \frac{du}{dt} + Au = F(u) + \dot{W}, \qquad u(0) = u_0 \in H,$$

where $\dot{w}$ is a white noise as the generalized temporal derivative of a Wiener process w with continuous trajectories on $\mathbb{R}$ and with values in $L_2(D)$. And F(u) is defined as the follow:

$$F(u) = \begin{pmatrix} -a + S_a(x,y) - b(y)S_o(x,y) + b(y)T(x,y) \\ -rq + PrRa\partial_y T - J(\psi, q + \beta y) \\ -J(T, \psi) \end{pmatrix}$$

Sufficient for this regularity is that the trace of the covariance is finite with respect to the space $L_2(D)$: $\mathrm{tr}_{L_2}Q < \infty$. In particular, we can choose the canonical probability space where the set of elementary events Ω consists of the paths of w and the probability measure $\mathbb{P}$ is the Wiener measure with respect to covariance Q.

In the following, we need a stationary Ornstein-Uhlenbeck process solving the linear stochastic equation on D

$$(8) \qquad \frac{dz}{dt} + A_1 z = \dot{w}$$

where $A_1 = -\Delta + (1 + b(y))$ is the linear operator with zero Neumann boundary condition at ∂D and with zero initial condition.

Lemma 3.1. *Suppose that the covariance Q has a finite trace : $\mathrm{tr}_{L_2}Q < \infty$. Then (8) has a unique stationary solution generated by*

$$(t, \omega) \to z(\theta_t\omega).$$

Moreover, $Z(\omega) = (z(\omega), 0, 0)$ is a random variable in V.

For the proof we refer to Da Prato and Zabczyk [17], Chapter 5, or Chueshov and Scheutzow [4].

For our calculations it will be appropriate to transform (7) into a differential equation without white noise but with random coefficients. We set

$$(9) \qquad\qquad\qquad v := u - Z$$

Thus we obtain a random differential equation in V'

$$(10) \qquad\qquad \frac{dv}{dt} + Av = F(v + Z(\theta_t\omega)), \qquad v(0) = v_0 \in H.$$

Equivalently, we can formulate the equation (10) using test functions

$$\frac{d}{dt}(v(t), \zeta) + a(v(t), \zeta) = (F(v(t) + Z(\theta_t\omega)), \zeta) \quad \text{for all } \zeta \in V.$$

We have obtained a differential equation without white noise but with random coefficients. Such a differential equation can be treated sample-wise for *any* sample ω. Hence it is simpler to consider (10) than to study the stochastic differential equation (7) directly. We are looking for solutions in

$$v \in C([0, \tau]; H) \cap L^2(0, \tau; V),$$

for all $\tau > 0$. If we can solve this equation then $u := v + Z$ defines a solution version of (7). For the well posedness of the problem we now have the following result.

Theorem 3.2. *(**Well-Posedness**) For any time $\tau > 0$, there exists a unique solution of (10) in $C([0, \tau]; H) \cap L_2(0, \tau; V)$. In particular, the solution mapping*

$$\mathbb{R}^+ \times \Omega \times H \ni (t, \omega, v_0) \to v(t) \in H$$

is measurable in its arguments and the solution mapping $H \ni v_0 \to v(t) \in H$ is continuous.

Proof. By the properties of A and F the random differential equation (10) is essentially similar to the 2 dimensional Navier Stokes equation. Hence we have existence and uniqueness and the above regularity assertions. □

On account of the transformation (9), we find that (7) also has a unique solution. Since the solution mapping

$$\mathbb{R}^+ \times \Omega \times H \ni (t, \omega, v_0) \to v(t, \omega, v_0) =: \varphi(t, \omega, v_0) \in H$$

is well defined, we can introduce a random dynamical system. On Ω we can define a shift operator θ_t on the paths of the Wiener process that pushes our noise:

$$w(\cdot, \theta_t\omega) = w(\cdot + t, \omega) - w(t, \omega) \quad \text{for } t \in \mathbb{R}$$

which is called the *Wiener shift*. Then $\{\theta_t\}_{t \in \mathbb{R}}$ forms a flow which is ergodic for the probability measure $\mathbb{P}$. The properties of the solution mapping cause the following relations

$$\varphi(t + \tau, \omega, u) = \varphi(t, \theta_\tau\omega, \varphi(\tau, \omega, u)) \quad \text{for } t, \tau \geq 0$$
$$\varphi(0, \omega, u) = u$$

for any $\omega \in \Omega$ and $u \in H$. This property is called the cocycle property of φ which is important to study the dynamics of random systems. It is a generalization of the semigroup property. The cocycle φ together with the flow θ forms a *random dynamical system*.

4. Dissipativity

In this section we are going to show that the coupled atmosphere-ocean system (1) is dissipative, in the sense that it has an absorbing (random) set. This definition has been used for deterministic systems [23]. This means that the solution vector v is contained in a particular region of the phase space H after a sufficiently long time. Dissipativity will be very important for understanding the asymptotic dynamics of the system. This dissipativity will give us estimate of the atmospheric temperature evolution under oceanic feedback. Dynamical properties that follow from this dissipativity will be considered in the next section. In particular, we will show that the coupled atmosphere-ocean system has a random attractor, has finite degree of freedom, and is ergodic under suitable conditions.

Let

$$(11) \qquad \tilde{v} = (\tilde{\Theta}, T), \quad \tilde{\Theta} = \Theta - z.$$

By integration by parts and Poincare inequality, we have

$$
(12) \qquad
\begin{aligned}
\frac{d}{dt}(\frac{\lambda_0}{2}\|\tilde{\Theta}\|_{L^2}^2 + \|T\|_{L^2}^2) \leq & -2\|\nabla\Theta\|_{L_2}^2 - \|\nabla T\|_{L^2}^2 \\
& -\frac{1}{2}\|\Theta\|_{L^2}^2 - \frac{\lambda_0}{2}\|T\|_{L^2}^2 + 6a^2 + 6\|S_a\|_{L^2}^2 + 6\|S_o\|_{L^2}^2.
\end{aligned}
$$

Here Poincare inequality $\lambda_0\|u\|^2 \leq \|\nabla u\|^2 (\lambda_0 > 0)$ is used. Here and in the following we stress that $0 < b(y) < 1$.
For q, we have

$$(13) \quad \frac{d}{2dt}\|q\|_{L^2}^2 = -\nu\|\nabla q\|_{L^2}^2 - r\|q\|_{L^2}^2 + PrRa\int_D \partial_y T q dx dy - \beta\int_D \frac{\partial\psi}{\partial x} q dx dy.$$

Under the following condition

$$(14) \qquad 4\nu r > \frac{\beta^2|l|^2}{\pi^2},$$

followed the argument of [12], we have for some positive constant $\alpha > 0$

$$\frac{d}{2dt}\|q\|_{L^2}^2 \leq -\alpha\|q\|_{L^2}^2 + PrRa\int_D \partial_y T q dx dy,$$

by the Cauchy-Schwarz inequality, we get

$$(15) \qquad \frac{d}{dt}\|q\|_{L^2}^2 \leq -\alpha\|q\|_{L^2}^2 + \frac{Pr^2Ra^2}{\alpha}\|\nabla T\|_{L^2}^2.$$

Collecting all these estimates, we have for some positive constants α_1 and C_1

$$\frac{d}{dt}(\frac{\lambda_0}{2}\|\tilde{\Theta}\|_{L^2}^2 + \|T\|_{L^2}^2 + \frac{\alpha\lambda_0}{2Pr^2Ra^2}\|q\|_{L^2}^2) + \alpha_2\|\nabla v\|^2$$

$$\leq -\alpha_1(\frac{\lambda_0}{2}\|\tilde{\Theta}\|_{L^2}^2 + \|T\|_{L^2}^2 + \frac{\alpha\lambda_0}{2Pr^2Ra^2}\|q\|_{L^2}^2) + C_1$$

By the Gronwall inequality, we finally conclude that

$$\|v\|_H^2 \leq C_2 \|v(0)\|_H^2 e^{-\alpha_1 t} + C_3, \tag{16}$$

and

$$\int_s^t \|\nabla v(\tau)\|_H^2 d\tau \leq C_4 \|v(0)\|_H^2 e^{-\alpha_1 s} + C_5(t-s), \text{ for any } 0 \leq s \leq t. \tag{17}$$

We now get the dissipativity of v. Roughly speaking dissipativity means that all trajectories of the system move to a bounded set in the phase space. For a random system we have the following version of dissipativity.

Definition 4.1. *A random set $B = \{B(\omega)\}_{\omega \in \Omega}$ consisting of closed bounded sets $B(\omega)$ is called absorbing for a random dynamical system φ if we have for any random set $D = \{D(\omega)\}_{\omega \in \Omega}, D(\omega) \in H$ bounded, such that $t \to \sup_{y \in D(\theta_t \omega)} \|y\|_H$ has a subexponential growth for $t \to \pm\infty$*

$$\varphi(t, \omega, D(\omega)) \subset B(\theta_t \omega) \quad \text{for } t \geq t_0(D, \omega) \tag{18}$$
$$\varphi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subset B(\omega) \quad \text{for } t \geq t_0(D, \omega).$$

B is called forward invariant if

$$\varphi(t, \omega, u_0) \in B(\theta_t \omega) \quad \text{if } u_0 \in B(\omega) \quad \text{for } t \geq 0.$$

From (16), we could get the existence of absorbing set $B(\omega) = \{v \in H, \|v\|_H^2 \leq 2C_3\}$. Suppose that $t \to \|v_0(\theta_{-t}\omega)\|_H^2$ growths not faster than subexponential. Then we have the subexponential growth for $B(\omega)$, and we get

Lemma 4.2. *The random set $B(\omega) = B(0, R(\omega)) = \{v \in H, \|v\|_H^2 \leq R = 2C_3\}$ is an absorbing and forward invariant set for the random dynamical system generated by (10).*

For the applications in the next section we need that the elements which are contained in the absorbing set satisfy a particular regularity. To this end we introduce the function space

$$\mathcal{H}^s := \{u \in H : \|u\|_s^2 := \|A^{\frac{s}{2}} u\|_H^2 < \infty\}$$

where $s \in \mathbb{R}$. The operator A^s is the s-th power of the positive and symmetric operator A. Note that these spaces are embedded in the Slobodeckij spaces H^s, $s > 0$. The norm of these spaces is denoted by $\| \cdot \|_{H^s}$. This norm can be found in Egorov and Shubin [10], Page 118. But we do not need this norm explicitly. We only mention that on $\mathcal{H}^s$ the norm $\| \cdot \|_s$ of H^s is equivalent to the norm of $\mathcal{H}^s$ for $0 < s$, see [15].

Our goal is it to show that $v(1, \omega, D)$ is a bounded set in $\mathcal{H}^s$ for some $s > 0$. This property causes the complete continuity of the mapping $v(1, \omega, \cdot)$. We now derive a differential inequality for $t\|v(t)\|_s^2$. By the chain rule we have

$$\frac{d}{dt}(t\|v(t)\|_s^2) = \|v(t)\|_s^2 + t\frac{d}{dt}\|v(t)\|_s^2.$$

Note that for the embedding constant $c_{6,s}$ between $\mathcal{H}^s$ and V

$$\int_0^t \|v\|_s^2 ds \leq c_{6,s}^2 \int_0^t \|v\|_V^2 ds \quad \text{for } s \leq 1$$

such that the left hand side is bounded if the initial conditions v_0 are contained in a bounded set in H. The second term in the above formula can be expressed as followed:

$$t\frac{d}{dt}(A^{\frac{s}{2}}v, A^{\frac{s}{2}}v)_H = 2t(\frac{d}{dt}v, A^sv)_H = -2t(Av, A^sv)_H + 2t(F(v + Z(\theta_t\omega)), A^sv)_H.$$

We have

$$(Av, A^sv)_H = \|A^{\frac{1}{2}+\frac{s}{2}}v\|_H = \|v\|_{1+s}^2.$$

For terms which including Jacobian operator, we apply some embedding theorems, see Temam [22] Page 12, then taking F_3 as a example we have for a constant $c_5 > 0$

$$(F_3(v), \zeta)_H \le c_6 \|v\|_{m_1+1}\|\psi\|_{m_2+1}\|\zeta\|_{m_3}, \quad \zeta \in H_{m_3}$$

where $m_1 + m_2 + m_3 \ge 1$ and $0 \le m_i < 1$. Here we use that D is of dimension 2. We then have for $m_1 = 0$, $m_2 = s < 1$ and $m_3 = 1 - s$

$$|(F_3(v), A^sT)_H| \le c_7\|T\|_V\|\psi\|_{1+s}\|T\|_{1+s}.$$

$\|\psi(t)\|_{1+s}$ is bounded by $c_1'\|q(t)\|_H$ by the definition of $\psi(t)$ and $\|v(t)\|_{L_\infty(0,T;H)} < \infty$. Hence we have for any $\varepsilon > 0$ a constant $c_8(\varepsilon)$:

$$(F_3(v(t)), A^sT(t))_H \le c_8(\varepsilon)\|q\|_{L_\infty(0,T;H)}^2\|T(t)\|_V^2 + \varepsilon\|T(t)\|_{1+s}^2$$

$$\le c_8(\varepsilon)\|v\|_{L_\infty(0,T;H)}^2\|v(t)\|_V^2 + \varepsilon\|v(t)\|_{1+s}^2,$$

where ε is chosen sufficiently small. The other terms could be estimated similarly, we omit it here. Using (17), we could obtain $\|v(t, \omega, v_0)\|_s$, $0 < s < \frac{1}{4}$ is bounded for $t_0 \le T < \infty, t_0 > 0$ if v_0 is contained in a bounded set. This allows us to write down the main assertion with respect to the dissipativity of this section.

Theorem 4.3. *For the random dynamical system generated by (10), there exists a compact random set $B = \{B(\omega)\}_{\omega \in \Omega}$ which satisfies Definition 4.1.*

We define

$$(19) \qquad B(\omega) = \overline{\varphi(1, \theta_{-1}\omega, B(0, R(\theta_{-1}\omega)))} \subset \mathcal{H}^s, \quad 0 < s < \frac{1}{4}.$$

In particular, $\mathcal{H}^s$ is compactly embedded in H.

5. RANDOM DYNAMICS

In this section we analyse the dynamical behavior of the coupled atmosphere-ocean system (1). However, it will be enough to analyse the transformed random dynamical system generated by (7). By the transformation (9) we can take over all these qualitative properties to the system (7).

We will consider following dynamical behavior: random attractors, atmospheric temperature evolution under oceanic feedback, and ergodicity.
We first consider random climatic attractors. We recall the following basic concept; see, for instance, Flandoli and Schmalfuß [11].

Definition 5.1. *Let φ be a random dynamical dynamical system. A random set $A = \{A(\omega)\}_{\omega \in \Omega}$ consisting of compact nonempty sets $A(\omega)$ is called random global attractor if for any random bounded set D we have for the limit in probability*

$$(\mathbb{P}) \lim_{t\to\infty} \text{dist}_H(\varphi(t, \omega, D(\omega)), A(\theta_t\omega)) = 0$$

and

$$\varphi(t, \omega, A(\omega)) = A(\theta_t \omega)$$

any $t \geq 0$ and $\omega \in \Omega$.

The essential long-time behavior of a random system is captured by a random attractor. In the last section we showed that the dynamical system φ generated by (7) is dissipative which means that there exists a random set B satisfying (18). In addition, this set is compact. We now recall and adapt the following theorem from [11].

Theorem 5.2. *Let φ be a random dynamical dynamical system on the state space H which is a separable Banach space such that $x \to \varphi(t, \omega, x)$ is continuous. Suppose that B is a set ensuring the dissipativity given in definition 4.1. In addition, B has a subexponential growth (see Definition 4.1) and is regular (compact). Then the dynamical system φ has a random attractor.*

This theorem can be applied to our random dynamical system φ generated by the stochastic differential equation (7). Indeed, all the assumptions are satisfied. The set B is defined in Theorem 4.3. Its subexponential growth follows from $B(\omega) \subset B(0, R(\omega))$ where the radius $R(\omega)$ has been introduced in the last section. Note that φ is a *continuous* random dynamical system; see Theorem 3.2. Thus φ has a random attractor. By the transformation (9), this is also true for the original coupled atmosphere-ocean system.

Corollary 5.3. *(***Random Attractor***) The coupled atmosphere-ocean system (1) has a random attractor.*

Now we consider random fixed point and ergodicity. We can do a small modification of (1). This modification is given when we replace ΔT by $\nu \Delta T$ where $\nu > 0$ is viscosity. Under particular assumptions about physical data in (1) , we can show that the behavior of our dynamical system is laminar. For a stochastic system, this means that after a relatively short time, all trajectories starting from different initial states show almost the same dynamical behavior. This can be seen easily if a, S_a and S_o are zero, there is no noise and ν is large. We will show that a laminar behavior also appears when a, S_a and S_o are small in some sense.
Mathematically speaking, laminar behavior means that a random dynamical system has a unique exponentially attracting random fixed point.

Definition 5.4. *A random variable $v^* : \Omega \to H$ is defined to be a random fixed point for a random dynamical system if*

$$\varphi(t, \omega, v^*(\omega)) = v^*(\theta_t \omega)$$

for $t \geq 0$ and $\omega \in \Omega$. A random fixed point v^* is called exponentially attracting if

$$\lim_{t \to \infty} \|\varphi(t, \omega, x) - v^*(\theta_t \omega)\|_H = 0$$

for any $x \in H$ and $\omega \in \Omega$.

Sufficient conditions for the existence of random fixed points are given in Schmalfuß [19]. We here formulate a simpler version of this theorem and it is appropriate for our system here.

Theorem 5.5. *(Random Fixed Point Theorem) Let φ be a random dynamical system and suppose that B is a forward invariant complete set. In addition, B has a subexponential growth, see Definition 4.1. Suppose that the following contraction conditions holds:*

$$(20) \qquad \sup_{v_1 \neq v_2 \in B(\omega)} \frac{\|\varphi(1, \omega, v_1) - \varphi(1, \omega, v_2)\|_H}{\|v_1 - v_2\|_H} \leq k(\omega)$$

where the expectation of $\log k$ denoted by $\mathbb{E} \log k < 0$. Then φ has a unique random fixed point in B which is exponentially attracting.

This theorem can be considered as a random version of the Banach fixed point theorem. The contraction condition is formulated in the mean for the right hand side of (20).

Theorem 5.6. *Assume that the physical data $|a|$, $\|S_a\|_{L^2}$, $\|S_o\|_{L^2}$ and the trace of the covariance for the noise $\mathrm{tr}_H Q$ are sufficiently small, and that the viscosity ν is sufficiently large. Then the random dynamical system generated by (7) has a unique random fixed point in B.*

Here we only give a short sketch of the proof. Let us suppose for a while that B is given by the ball $B(0, R)$ introduced in Lemma 4.2. Suppose that the data in the assumption of the lemma are small and ν is large. Then it follows that $\mathbb{E}R$ is also small. To calculate the contraction condition we have to calculate $\|\varphi(1, \omega, v_1(\omega)) - \varphi(1, \omega, v_2(\omega))\|_H^2$ for arbitrary random variables v_1, $v_2 \in B$. By the property of Jacobian operator we have that

$$\langle J(q_1, \psi_1) - J(q_1, \psi_1), q_1 - q_2 \rangle \leq c_9 \|q_1 - q_2\|_{W_2^1}^2 + c_{10} \|q_1\|_{H^1(D)}^2 \|q_1 - q_2\|_{L_2}^2$$

where the constant c_{10} can be chosen sufficiently small if ν is large. By direct estimate, we get

$$\frac{d}{dt} \|\varphi(t, \omega, v_1(\omega)) - \varphi(t, \omega, v_2(\omega))\|_H^2$$
$$\leq (-\alpha' + c_{10} \|\varphi(t, \omega, v_2(\omega))\|_V^2) \|\varphi(t, \omega, v_1(\omega)) - \varphi(t, \omega, v_2(\omega))\|_H^2$$

for some positive α' depending on ν. From this inequality and the Gronwall lemma it follows that the contraction condition (20) is satisfied if

$$\mathbb{E} \sup_{u_2 \in B(\omega)} c_{10} \int_0^1 \|\varphi(t, \omega, v_2)\|_V^2 dt < \alpha'.$$

But by the energy inequality this property is satisfied if the $\mathbb{E}R$ and $\mathbb{E}\|z\|_V^2$ is sufficiently small which follows from the assumptions.

Let now B be the random set defined in (19). Since the set B introduced in (19) is absorbing any state the fixed point v^* is contained in this B. In addition v^* attracts *any* state from H and not only states from B.

Corollary 5.7. *(Unique Random Fixed Point) Assume that the physical data $|a|$, $\|S_a\|_{L^2}$, $\|S_o\|_{L^2}$ and the trace of the covariance for the noise $\mathrm{tr}_H Q$ are sufficiently small, and that the viscosity ν is sufficiently large. Then, through the transformation (9), the original system (1) has a unique exponentially attracting random fixed point $u^*(\omega) = v^*(\omega) + Z(\omega)$, where $u = (\Theta, q, T)$.*

The uniqueness of this random fixed point implies *ergodicity*. We will comment on this issue at the end of this section.

By the well-posedness Theorem 3.2, we know that the stochastic differential equation (7) for the coupled atmosphere-ocean system has a unique solution. The solution is a Markov process. We can define the associated Markov operators $\mathcal{T}(t)$ for $t \geq 0$, as discussed in [21, 20]. Moreover, $\{\mathcal{T}(t)\}_{t\geq 0}$ forms a semigroup.

Let M^2 be the set of probability distributions μ with finite energy, i.e.,

$$\int_H \|u\|_H^2 d\mu(u) < \infty.$$

Then the distribution of the solution $u(t)$ (at time t) of the stochastic differential equation (7) is given by

$$\mathcal{T}(t)\mu_0,$$

where the distribution μ_0 of the initial data is contained in M^2.

We note that the expectation of the solution $\|u(t)\|_H^2$ can be expressed in terms of this distribution $\mathcal{T}(t)\mu_0$:

$$\mathbb{E}\|u(t)\|_H^2 = \int_H \|u\|_H^2 d\mathcal{T}(t)\mu_0.$$

We can derive the following energy inequality in the mean, using our earlier estimates in (16) and (17):

Theorem 5.8. *The dynamical quantity $u = (\Theta, q, T)$ of the coupled atmospheric-ocean system(1) satisfy the estimate*

$$\mathbb{E}\|u(t)\|_H^2 + \alpha\mathbb{E}\int_0^t \|u(\tau)\|_V^2 d\tau \leq \mathbb{E}\|u_0\|_H^2 + t\,c_{11} + t\,\mathrm{tr}_{L_2}Q,$$

where the positive constants c_{11} and α depend on physical data a, $\|S_a\|_{L^2}$, $\|S_o\|_{L^2}$, Pr and Ra.

By the Gronwall inequality, we further obtain the following result about the asymptotic mean-square estimate.

Corollary 5.9. *(**Asymptotic Mean-Square Estimate**) For the expectation of the dynamical quantity $u = (\Theta, q, T)$ of the coupled atmospheric-ocean system (1), we have the asymptotic estimate*

$$\limsup_{t\to\infty}\mathbb{E}\|u(t)\|_H^2 = \limsup_{t\to\infty}\int_H \|u\|_H^2 d\mathcal{T}(t)\mu_0 \leq \frac{c_{11} + \mathrm{tr}_{L_2}Q}{c_{12}}$$

if the initial distribution μ_0 of the random initial condition $u_0(\omega)$ is contained in M^2. Here $c_{12} > 0$ also depends on physical data. In particular, we have asymptotic mean-square estimate for the atmospheric temperature evolution under oceanic feedback

$$(21) \qquad \limsup_{t\to\infty}\mathbb{E}\|\Theta\|_H^2 \leq \frac{c_{11} + \mathrm{tr}_{L_2}Q}{c_{12}}.$$

Thus the atmospheric temperature $\Theta(y, t)$, as modeled by the coupled atmosphere-ocean system (1), is bounded asymptotically in mean-square norm in terms of physical quantities such as the trace of the covariance operator of the external noise, the earth's longwave radiative cooling coefficient a, and the empirical functions $\|S_a\|_{L^2}$ and $\|S_o\|_{L^2}$ representing the latitudinal dependence of the shortwave solar radiation, as well as the Prandtl number Pr and the Rayleigh number Ra for oceanic fluids.

By the estimates of Theorem 5.8, we are able to use the well known Krylov-Bogolyubov procedure to conclude the existence of invariant measures of the Markov semigroup.

Corollary 5.10. *The semigroup of Markov operators* $\{\mathcal{T}(t)\}_{t \geq 0}$ *possesses an invariant distribution* μ_i *in* M^2:

$$\mathcal{T}(t)\mu_i = \mu_i \quad \text{for } t \geq 0.$$

In fact, the limit points of

$$\left\{ \frac{1}{t} \int_0^t \mathcal{T}(\tau)\mu_0 d\tau \right\}_{t \geq 0}$$

for $t \to \infty$ are invariant distributions. The existence of such limit points follows from the estimate in Theorem 5.8.

In some situations, the invariant measure may be unique. For example, the unique random fixed point in Corollary 5.7 is defined by a random variable $u^*(\omega) = v^*(\omega) + z(\omega)$. This random variable corresponds to a unique invariant measure of the Markov semigroup. More specifically, this unique invariant measure is the expectation of the Dirac measure with the random variable as the random mass point

$$\mu_i = \mathbb{E}\delta_{u^*(\omega)}.$$

Because the uniqueness of invariant measure implies ergodicity [18], we conclude that the coupled atmosphere-ocean model (1) is ergodic under the suitable conditions in Corollary 5.7 for physical data and random noise. We reformulate Corollary 5.7 as the following ergodicity principle.

Theorem 5.11. (**Ergodicity**) *Assume that the physical data* $|a|$, $\|S_a\|_{L^2}$, $\|S_o\|_{L^2}$ *and the trace of the covariance for the noise* $\mathrm{tr}_H Q$ *are sufficiently small, and that the viscosity* ν *is sufficiently large. Then the coupled atmosphere-ocean system (1) is ergodic, namely, for any observable of the coupled atmosphere-ocean flows, its time average approximates the statistical ensemble average, as long as the time interval is sufficiently long.*

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(A. Du and J. Duan) Department of Applied Mathematics, Illinois Institute of Technology, Chicago, IL 60616, USA
E-mail address: duaijun@iit.edu; duan@iit.edu

(H. Gao) Department of Mathematics, Nanjing Normal University, Nanjing 210097, China
E-mail address, H. Gao: gaohj@njnu.edu.cn

(T. Özgökmen) RSMAS/MPO, University of Miami, Miami, Florida, USA
E-mail address, T. Özgökmen: tamay@rsmas.miami.edu