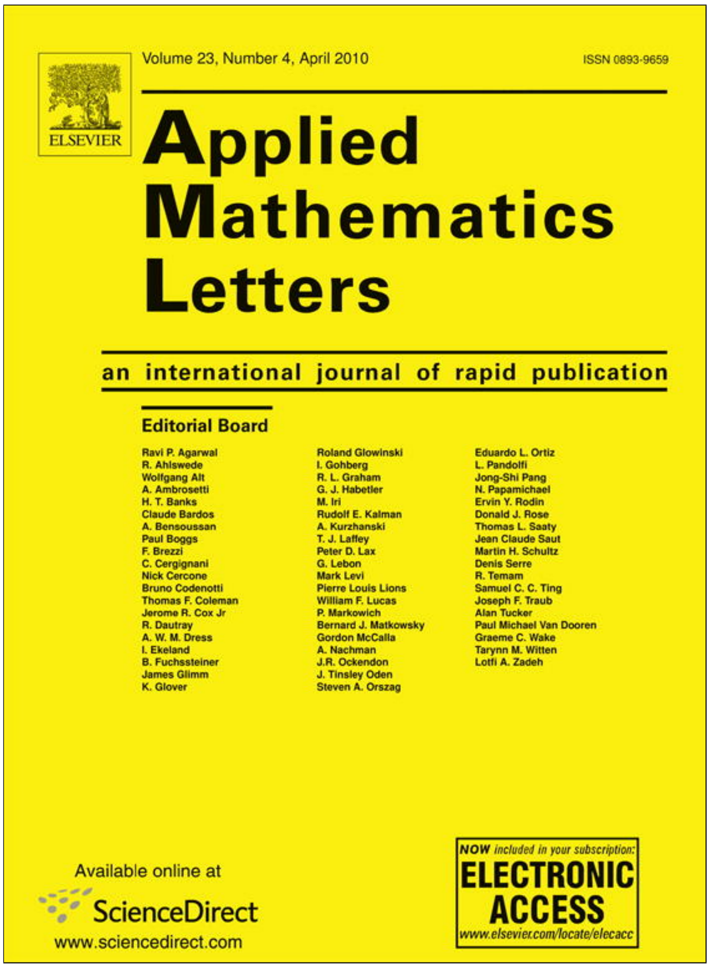
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Bridging the Boussinesq and primitive equations through spatio-temporal filtering

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ABSTRACT

We propose a novel approach for bridging the Boussinesq equations and the primitive equations. This approach uses spatio-temporal filtering as an alternative to traditional scaling arguments.

1. Introduction

One of the starting points in the derivation of simplified models for geophysical flows is represented by the *Boussinesq equations* (BE) [1–3]:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + f\mathbf{k} \times \mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu \Delta \mathbf{u} - \frac{\rho}{\rho_0} g\mathbf{k} \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

$$\frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla)\rho = \kappa \Delta \rho, \tag{3}$$

where $\mathbf{u} = (u, v, w)$ is the fluid velocity, p the pressure, ρ_0 the reference density, ρ the density perturbation, ν the kinematic viscosity, κ the molecular diffusivity, g the gravitational acceleration, $\mathbf{k}$ the unit vector in the vertical direction, and f the Coriolis parameter. For many realistic geophysical flows, the numerical discretization of the BE yields a prohibitively high computational cost. Thus, a significant research effort has been directed at generating mathematical models that are more computationally efficient than the BE, yet physically accurate [4]. The tool of choice in generating these simplified models has been *scaling*. In this note, we put forth *spatio-temporal filtering* as an alternative methodology.

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2. Scaling

We now illustrate how the BE are simplified through scaling. Let Ω be the angular rate of rotation. We choose the following scales: L the horizontal dimension of the domain, H the vertical dimension, U the horizontal velocity, W the vertical velocity, $T_z = H/W$ the vertical time scale, $T_{xy} = L/U$ the horizontal time scale, and P the pressure scale.

Theorem 2.1. *If the following assumptions are satisfied*

$$\frac{1}{\Omega} \leq T_{xy} \leq T_z \tag{4}$$

$$H \ll L \tag{5}$$

$$\frac{\Omega U}{g \frac{\Delta\rho}{\rho_0}} \ll 1 \tag{6}$$

$$\frac{\frac{\nu U}{H^2}}{\Omega U} \leq 1, \tag{7}$$

then the Boussinesq equations (1)–(3) *reduce to the primitive equations* (PE)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2} \tag{8}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2} \tag{9}$$

$$0 = -\frac{\partial p}{\partial z} - g\rho \tag{10}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{11}$$

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \kappa\frac{\partial^2 \rho}{\partial z^2}. \tag{12}$$

Proof. The proof is somehow different from the usual arguments [1], since it centers around the time scale defined in (4). We only prove the claim for the vertical momentum equation

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - f_* u = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_0} + \nu\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial z^2} \tag{13}$$

since the other equations can be treated similarly.

First, by (4) and (5), we have $W \ll U$. Furthermore, by (4), we have $\frac{W}{T_z} \leq \Omega W$. These two inequalities imply that the first term in (13) is dominated by the fifth term, and thus can be dropped.

By (4), we have $\frac{U}{L} \leq \Omega$. This inequality together with $W \ll U$ imply $\frac{W}{L} \ll \Omega$, which in turn implies that the second and third terms in (13) are dominated by the fifth term, and thus can be dropped.

By (5), we have $\frac{W}{H} \leq \frac{U}{L}$, which implies $W\frac{W}{H} \leq U\frac{W}{L}$. This inequality together with $\frac{W}{L} \ll \Omega$ imply that the fourth term in (13) is dominated by the fifth term, and thus can be dropped.

Assumption (5) implies that the third and fourth terms on the RHS of (13) are much smaller than the very last term, so they can be dropped.

Assumption (6) implies that the last term on the LHS of (13) is much smaller than the second term on the RHS, so it can be dropped.

Finally, inequality $W \ll U$ and assumption (7) imply that the last term on the RHS of (13) is much smaller than the last term on the LHS, so it can be dropped. Thus, (3) reduces to the *hydrostatic approximation* (10). □

3. Spatio-temporal filtering

The main advantage of the PE over the BE is their increased computational efficiency. This, however, comes at a price: because of the hydrostatic approximation, the PE are not as accurate as the BE. Thus, a significant research effort has been directed at devising *intermediate models*, i.e., models that are more efficient than the BE, yet more accurate than the PE [4].

We propose the use of *spatio-temporal filtering* to develop such intermediate models. We start by considering a spatio-temporal filter $g_{(\delta\mathbf{x},\delta t)}(\mathbf{x}, t)$, where $\delta\mathbf{x} = (\delta x, \delta y, \delta z)$ is the spatial radius and δt is the temporal radius. We then convolve the flow variables with this spatio-temporal filter

$$\overline{\mathbf{u}}(\mathbf{x}, t) = (g_{(\delta\mathbf{x},\delta t)} * \mathbf{u})(\mathbf{x}, t) = \int_0^t \int_\Omega g_{(\delta\mathbf{x},\delta t)}(\mathbf{x} - \mathbf{x}', t - t')\mathbf{u}(\mathbf{x}', t')\, d\mathbf{x}'\, dt'.$$

This averaging procedure, similar to that used in large eddy simulation of turbulent flows [5], eliminates the flow structures that occur at spatial and temporal scales larger than $\delta\mathbf{x}$ and δt, respectively. To obtain equations for the averaged flow variables, we convolve the BE with the spatio-temporal filter $g_{(\delta\mathbf{x},\delta t)}$ and obtain the *filtered Boussinesq equations* (FBE)

$$\frac{\partial \overline{\mathbf{u}}}{\partial t} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}} + \nabla \cdot \boldsymbol{\tau} + f\mathbf{k} \times \overline{\mathbf{u}} = -\frac{1}{\rho_0}\nabla \overline{p} + \nu \Delta \overline{\mathbf{u}} - \frac{\overline{\rho}}{\rho_0} g\mathbf{k} \tag{14}$$

$$\nabla \cdot \overline{\mathbf{u}} = 0 \tag{15}$$

$$\frac{\partial \overline{\rho}}{\partial t} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\rho} + \nabla \cdot \boldsymbol{\sigma} = \kappa \Delta \overline{\rho}, \tag{16}$$

where $\boldsymbol{\tau} = \overline{\mathbf{u}\mathbf{u}} - \overline{\mathbf{u}}\,\overline{\mathbf{u}}$ and $\boldsymbol{\sigma} = \overline{\mathbf{u}\rho} - \overline{\mathbf{u}}\,\overline{\rho}$.

We first note that numerical simulations with the FBE are dramatically more efficient than those with the BE. Indeed, instead of resolving all the features in the flow variables, in the FBE only the spatio-temporal averages are approximated.

Second, the FBE represent a more accurate physical description of the flow than the PE. Indeed, vertical motion in the FBE is significantly more accurate than that in the PE. Furthermore, nonlinear interactions are included in FBE through $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, whereas in the PE only the leading order approximations of the nonlinear terms are considered.

The above two remarks indicate that the FBE represent an intermediate model, i.e., a model whose computational efficiency and physical accuracy are between those of the BE and the PE. The next result shows that the FBE actually represent a *bridging mechanism* between the BE and the PE.

Theorem 3.1. *Let* $\Delta z := \sqrt{\frac{\nu}{\Omega}}$, $\Delta x = \Delta y = \frac{L}{H}\Delta z$, *and* $\Delta t := \frac{1}{\Omega}$. *When* $(\delta\mathbf{x}, \delta t) \to (\mathbf{0}, 0)$, *the FBE are asymptotically equivalent to the BE. When* $(\delta\mathbf{x}, \delta t) \to (\Delta x, \Delta y, \Delta z, \Delta t)$ *and assumptions* (5) *and* (6) *are satisfied, the FBE are asymptotically equivalent to the PE.*

Proof. The first statement is straightforward. Indeed, when $(\delta\mathbf{x}, \delta t) \to (\mathbf{0}, 0)$, the spatio-temporal filtering reduces to convolution with a delta function, and thus the averaged flow variables converge to the original (unfiltered) variables: e.g., $\overline{\mathbf{u}} \to \mathbf{u}$. Furthermore, $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ vanish, and thus the FBE reduce to the BE.

The second statement follows by noticing that the vertical length scale Δz and time scale Δt are naturally yielded by assumptions (4) and (7), respectively. Since assumptions (5) and (6) are also valid, Theorem 2.1 proves the second statement. □

4. Approximate deconvolution models

In order to use the FBE in the numerical simulations of geophysical flows, one needs to close the system, i.e., to model $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ in terms of the filtered variables $\overline{\mathbf{u}}$ and $\overline{\rho}$. We propose the use of *approximate deconvolution* (AD), which is a closure approach derived solely on mathematical grounds, without any phenomenology (such as the energy cascade). The AD procedure is based on the van Cittert decomposition and has been proposed in the LES context in [6] and later adapted to temporal filtering in [7,8].

The AD approach is based on the following idea: Given approximations for $\overline{\mathbf{u}}$ and $\overline{\rho}$, find approximations for $\mathbf{u}$ and ρ, and use them to approximate $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ (and thus close the FBE):

$$\tau_{ij} := \overline{u_i u_j} - \overline{u}_i\overline{u}_j \approx \overline{v_i v_j} - \overline{v}_i\overline{v}_j, \qquad \sigma_j := \overline{\rho u_j} - \overline{\rho}\,\overline{u}_j \approx \overline{\theta v_j} - \overline{\theta}\,\overline{v}_j, \tag{17}$$

where v_j and θ are the approximately deconvolved (or defiltered) velocity and density, respectively, achieved by repeated filtering

$$v_i = \sum_{n=0}^{N} a_n \left(G^n u_i\right), \qquad \theta_i = \sum_{n=0}^{N} b_n \left(G^n \rho_i\right). \tag{18}$$

The operator G corresponds to the spatio-temporal filtering (e.g., $G\mathbf{u} = g_{(\delta\mathbf{x},\delta t)} * \mathbf{u}$) and the coefficients c_n, d_n can be chosen to minimize phase error effects associated with the spatio-temporal filtering. For the computational implementation of the filtering, we can use a differential filter [7,8,5].

5. Conclusions

We have proposed a new paradigm for the development of simplified computational models for geophysical flows. This paradigm is centered around spatio-temporal filtering. The novelty of this approach is that it allows a consistent treatment of all flow scales, from the smallest to the largest. Thus, the same mathematical model can tackle geophysical flows taking place at completely different spatial and temporal scales by simply changing the radii of the spatio-temporal filter according to the available computational resources. Finally, we have also used approximate deconvolution to close the FBE and yield an accurate yet computationally tractable model. We plan to thoroughly investigate both the achievements and limitations of the proposed methodology in future theoretical and computational studies.

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